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THE STABILITY OF THE PLANE MOTIONS OF A RIGID BODY IN THE KOVALEVSKAYA CASE†

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The problem of the orbital stability of the pendulum-like oscillations and rotations of a heavy rigid body with one fixed point is solved in the Kovalevskaya case. © 2001 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM AND THE MAIN RESULT

Consider the motion of a rigid body about a fixed point O in a uniform gravity field. We will refer the motion to a fixed system of coordinates $OXYZ$, the OZ axis of which is directed vertically upwards. We connect with the rigid body a system of coordinates $Oxyz$, formed by the principal axes of inertia of the body for the point O. We denote the moments of inertia of the body about the Ox, Oy and Oz axes by A, B and C respectively. Suppose mg is the weight of the body, l is the distance from the centre of gravity to the fixed point O, and x_*, y_*, z_* are the coordinates of the centre of gravity in the $Oxyz$ system. In the Kovalevskaya case $A = B = 2C, z_* = 0$. Without loss of generality we can assume $x_* = l, y_* = 0$.

We will use the Euler angles ψ, θ, φ as the generalized coordinates, which specify the orientation of the body in space. The equations of motion have the form [1]

$$2\frac{dp}{dt} - qr = 0, \quad 2\frac{dq}{dt} + rp = \lambda^2\gamma_3, \quad \frac{dr}{dt} = -\lambda^2\gamma_2$$
$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3, \quad \frac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_1, \quad \frac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2, \tag{1.1}$$
$$p = \frac{d\psi}{dt}\gamma_1 + \frac{d\theta}{dt}\cos\varphi, \quad q = \frac{d\psi}{dt}\gamma_2 - \frac{d\theta}{dt}\sin\varphi, \quad r = \frac{d\psi}{dt}\gamma_3 + \frac{d\varphi}{dt}$$
$$\gamma_1 = \sin\theta\sin\varphi, \qquad \gamma_2 = \sin\theta\cos\varphi, \qquad \gamma_3 = \cos\theta$$

In (1.1) we have introduced the notation $\lambda^2 = mgl/C$.

In addition to the three algebraic first integrals, which always exist in the case of the motion of a heavy rigid body around a fixed point – the energy integral, the integral of areas and the integral $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$, Eqs (1.1) allow of a fourth algebraic integral (the Kovalevskaya integral)

$$(p^2 - q^2 - \lambda^2\gamma_1)^2 + (2pq - \lambda^2\gamma_2)^2 = \text{const} \tag{1.2}$$

In this case Eqs (1.1) are integrable [3]. But since their solution has a quite complex mathematical form, it is of interest to investigate special solutions.

Equations (1.1) have solutions for which $\psi = 0$, $\theta = \pi/2$. In this case

$$p = q = 0, r = d\varphi/dt, \gamma_1 = \sin\varphi, \gamma_2 = \cos\varphi, \gamma_3 = 0.$$

For these solutions the constant area integral (the projection of the kinetic moment of the body onto the vertical) is equal to zero, the axis of symmetry of the body Oz is fixed and occupies a horizontal position, while the motion of the body around this axis is described by the differential equation of a mathematical pendulum. In general, this motion is rotation with an angular velocity of arbitrary value, which varies periodically with time, or oscillations of arbitrary amplitude. The purpose of this paper is to solve the problem of the orbital stability of these pendulum-like motions of the body. Here we will assume that the projection of the kinetic moment onto the vertical is unperturbed, i.e. it is assumed to

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be equal to zero in the perturbed motion also. The main result of this paper is a proof of the following theorem.

Theorem 1. If the amplitude of plane oscillations of a Kovalevskaya top around the axis of dynamic symmetry does not exceed $\pi/2$, these oscillations are orbitally stable; if the amplitude is greater than $\pi/2$, there will be instability. The plane rotations are always unstable.

2. THE HAMILTION FUNCTION. DESCRIPTION OF THE UNPERTURBED MOTION

Suppose $p_\psi, p_\theta, p_\varphi$ are generalized momenta, corresponding to the Euler angles. The angle ψ is a cyclic coordinate and p_ψ is the projection of the kinetic moment onto the vertical, which has a constant value. By assumption, $p_\psi = 0$. We will put

$$\varphi = 3\pi/2 + q_1, \qquad \theta = \pi/2 + q_2, \qquad p_\varphi = C\lambda p_1, \qquad p_\theta = C\lambda p_2$$

and introduce the dimensionless time $\tau = \lambda t$. Then, the motion of the rigid body in the Kovalevskaya case can be described (when $p_\psi = 0$) by canonical equations, specified by the dimensionless Hamiltonian

$$H = \tfrac{1}{2} p_1^2 - \cos q_1 \cos q_2 + \tfrac{1}{4} p_2^2 + \tfrac{1}{4} p_1^2 \operatorname{tg}^2 q_2 \qquad (2.1)$$

The Kovalevskaya integral (1.2) can be represented in the form

$$[(p_1^2 \operatorname{tg}^2 q_2 - p_2^2) + 4\cos q_1 \cos q_2]^2 + 4[p_1 p_2 \operatorname{tg} q_2 - 2\sin q_1 \cos q_2]^2 = \text{const} \qquad (2.2)$$

The solution in which $q_2 = p_2 = 0$ corresponds to plane motions of the body, while the variables q_1 and p_1 are described by equations with Hamiltonian

$$H^{(0)} = \tfrac{1}{2} p_1^2 - \cos q_1 \qquad (2.3)$$

These equations have the integral $H^{(0)} = h = \text{const}$. When $-1 < h < 1$ the body executes plane oscillations in the neighbourhood of a stable equilibrium position, for which the centre of gravity of the body lies on the vertical OZ below the fixed point O. When $h > 1$ plane rotations occur: the angles φ of rotation of the body about its horizontal axis Oz increases or decreases monotonically with time.

For the purposes of further investigation of the stability of the plane motions, it is more convenient to write Hamiltonian (2.3) in action-angle variables I, w [1]. In the case of oscillations we put $k_1 = \sin(\beta/2)$, where β is the oscillation amplitude ($0 < \beta < \pi$). The canonical univalent replacement of variables $q_1, p_1 \to I, w$ is given by the equalities

$$q_1 = 2\arcsin[k_1 \operatorname{sn}(u, k_1)], \quad p_1 = 2k_1 \operatorname{cn}(u, k_1), \quad u = 2\pi^{-1} K(k_1) w \qquad (2.4)$$

Here $k_1 = k_1(I)$ is a function which is the inverse of the function

$$I = 8\pi^{-1}[E(k_1) - (1 - k_1^2) K(k_1)] \qquad (2.5)$$

In (2.4) and (2.5) and below we use the standard notation for elliptic functions and integrals [3]. The oscillation frequency is calculated from the formula

$$\omega_1 = \pi/(2K(k_1)) \qquad (2.6)$$

In I, w variables the Hamilton function (2.3) takes the form

$$H^{(0)} = 2k_1^2 - 1 \qquad (2.7)$$

In the case of rotations we put $k_2^2 = 2(1 + h)^{-1}$. The variables I, w are introduced by the formulae

$$q_1 = 2\operatorname{am}(u, k_2), \quad p_1 = 2k_2^{-1} \operatorname{dn}(u, k_2), \quad u = \pi^{-1} K(k_2) w \qquad (2.8)$$

Here $k_2 = k_2(I)$ is a function which is the inverse of the function

$$I = 4\pi^{-1}k_2^{-1}E(k_2) \tag{2.9}$$

For the rotation frequency we have the expression

$$\omega_2 = \pi/(k_2K(k_2)) \tag{2.10}$$

In I, w variables Hamilton function (2.3) has the form

$$H^{(0)} = 2k_2^{-2} - 1 \tag{2.11}$$

In unperturbed motion we have $q_2 = p_2 = 0, I = I_0 =$ const, and the variables q_1 and p_1, for a specified permissible value of I_0, are defined by (2.4)–(2.6) in the case of oscillations and by (2.8)–(2.10) in the case of rotations. In this case $w = \omega_1\tau + w(0)$ in the case of oscillations and $w = \omega_2\tau + w(0)$ in the case of rotations.

We will introduce a perturbation of the action variable $r_1 = I - I_0$. The problem of the orbital stability of the plane oscillations and rotations of a body is equivalent to the problem of their stability with respect to the variables q_2, p_2 and r_1.

3. INVESTIGATION OF THE LINEARIZED EQUATIONS OF PERTURBED MOTION

From (2.1) and (2.3)–(2.11) we obtain the part H_2 of the Hamiltonian of the perturbed motion that is quadratic in $q_2, p_2, |r_1|^{1/2}$

$$H_2 = \omega r_1 + \tfrac{1}{4}p_2^2 + \tfrac{1}{4}(p_1^2 + 2\cos q_1)q_2^2 \tag{3.1}$$

In (3.1) the quantities ω, q_1 and p_1 correspond to the unperturbed motion and are defined for $I = I_0$ by formulae (2.6) and (2.4) in the case of oscillations, and by formulae (2.10) and (2.8) in the case of rotations.

In the linearized equations of the perturbed motion $r_1 =$ const, and the change in the variables q_2 and p_2, if we take w as the independent variables, is described by the equations

$$\frac{dq_2}{dw} = \frac{\partial h_2}{\partial p_2}, \quad \frac{dp_2}{dw} = -\frac{\partial h_2}{\partial q_2} \tag{3.2}$$

where

$$h_2 = \frac{1}{4\omega}[p_2^2 + (p_1^2 + 2\cos q_1)q_2^2] \tag{3.3}$$

The Kovalevskaya integral (2.2) for the equations of perturbed motion can be written using the following series in powers of q_2, p_2 and r_1

$$K = k_2 + k_4 + \ldots + k_n + \ldots = \text{const} \tag{3.4}$$

where k_n is the form of the power of n of $q_2, p_2, |r_1|^{1/2}$ with coefficients which depend on q_1 and p_1, which correspond to the unperturbed motion. In this case

$$k_2 = (p_1^2\cos q_1 - 2)q_2^2 - 2p_1\sin q_1\, q_2p_2 - \cos q_1 p_2^2 \tag{3.5}$$

It can be shown by a direct check that k_2 is the first integral of linear equations (3.2).

Suppose $\mathbf{X}(w)$ is the matrix of the fundamental solutions of system (3.2), normalized by the condition $\mathbf{X}(0) = \mathbf{E}$, where $\mathbf{E}$ is the second-order identity matrix. The elements $x_{ij}(w)$ of the matrix $\mathbf{X}$ satisfy the equations

$$\frac{dx_{1j}}{dw} = \frac{1}{2\omega}x_{2j}, \quad \frac{dx_{2j}}{dw} = -\frac{1}{2\omega}(p_1^2 + 2\cos q_1)x_{1j}; \quad j = 1, 2 \tag{3.6}$$

and initial conditions

$$x_{11}(0)=x_{22}(0)=1,\quad x_{12}(0)=x_{21}(0)=0 \tag{3.7}$$

The right-hand sides of Eqs (3.6) have period. T with respect to w, and, as can be seen from (2.4) and (2.8), $T = \pi$ in the case of oscillations and $T = 2\pi$ in the case of rotations. In view of the Hamiltonian form of system (3.6) we have the following equality for any w

$$x_{11}x_{22}-x_{21}x_{12}=1 \tag{3.8}$$

The characteristic equation of the matrix $\mathbf{X}(T)$ can be written in the form

$$\rho^2-2a\rho+1=0 \tag{3.9}$$

where $2a = x_{11}(T) + x_{22}(T)$. If $|a| < 1$, the roots of Eq. (3.9) are equal in modulus to unity and are different. In this case there is stability [4] in the linear approximation. If $|a| > 1$, Eq. (3.9) has a root, whose modulus is greater than unity and the periodic motion considered is unstable, not only in the linear approximation but also within the scope of the complete non-linear equations of the perturbed motion [4].

By relations (3.2), (3.5) and (3.7) the solutions of Eqs (3.6) for any value of w satisfy the equations

$$(p_1^2\cos q_1-2)x_{1j}^2-2p_1\sin q_1x_{1j}x_{2j}-\cos q_1x_{2j}^2=c_j=\text{const},\quad j=1,2 \tag{3.10}$$
$$c_1=p_1^2(0)\cos q_1(0)-2,\quad c_2=-\cos q_1(0)$$

Here $q_1(0)$ and $p_1(0)$ are the values of the functions q_1 and p_1 when $w = 0$, calculated from (2.4) in the case of oscillations and from (2.8) in the case of rotations. Putting $w = T$ in (3.10) we obtain that the following equalities hold

$$2(2b^2-1)x_{11}^2(T)-x_{21}^2(T)=2(2b^2-1),\quad 2(2b^2-1)x_{12}^2(T)-x_{22}^2(T)=-1 \tag{3.11}$$

where $b = k_1$ in the case of oscillations and $b = k_2^{-1}$ in the case of rotations.

From (3.11) and (3.8) we obtain that $x_{11}(T) = x_{22}(T)$, and, consequently, the value of a in Eq. (3.9) satisfies the equality

$$a^2=x_{22}^2(T)=1+2(2b^2-1)x_{12}^2(T) \tag{3.12}$$

By considering the asymptotic form of the solutions of Eqs (3.6) we can obtain that in the case of small-amplitude oscillations ($k_1 \to 0$) we have $x_{12}(w) \to (\sqrt{2}/2)\sin(\sqrt{2}w/2)$, while in the case of rapid rotations ($k_2 \to 0$) we have $x_{12}(w) \to 1/2k_2 \sin(w/2)$. A numerical analysis showed that $x_{12}(T)$ is a monotonically increasing function (of k_1 in the case of oscillations and of k_2 in the case of rotations). Hence, the quantity $x_{12}(T)$ in Eq. (3.12) is always non-zero: in the case of oscillations $x_{12}(T) > (\sqrt{2}/2)\sin(\pi\sqrt{2}/2)$, while in the case of rotations $x_{12}(T) > 0$.

Since in the case of rotations the quantity $b > 1$, it can be immediately seen from (3.12) that $|a| > 1$. Consequently, plane rotations of a Kovalevskaya top around the axis of dynamic symmetry are always orbitally unstable.

Oscillations as a function of their amplitude β (i.e. as a function of the maximum deviation of the Ox axis of the body, on which the centre of gravity lies, from its stable equilibrium position along the vertical) can be orbitally stable or unstable. It can be seen from (3.12) that if $\beta > \pi/2$ (i.e. $\sqrt{2}/2 < k_1 < 1$), then $|a| > 1$. Hence, oscillations with an amplitude exceeding $\pi/2$ are orbitally unstable. If the oscillation amplitude does not exceed $\pi/2$ (i.e. $0 < k_1 \leq \sqrt{2}/2$), then $|a| \leq 1$ and the linear approximation is insufficient to enable us to draw rigorous conclusions regarding their orbital stability.

4. NON-LINEAR ANALYSIS OF THE STABILITY OF THE OSCILLATIONS IN THE CASE WHEN $\beta \leq \pi/2$

We will take $\omega_1\tau$ as the independent variable. From (2.4)–(2.7), using the well-known rules for actions with elliptic functions and integrals [3], we can obtain the Hamiltonian of the perturbed motion in the form of the following series

$$H=r_1+h_2+H_4+\dots \tag{4.1}$$

The quantity h_2 is specified by (3.3) in which $\omega = \omega_1$, while q_1 and p_1 correspond to unperturbed motion. The function H_4 is defined by the equations

$$H_4 = \omega_1^{-1}(f_0 r_1^2 + \tfrac{1}{24} f_1 q_2^4 + \tfrac{1}{4} f_2 r_1 q_2^2) \tag{4.2}$$

$$f_0 = -\frac{\pi^2[E(k_1) - (1-k_1^2)K(k_1)]}{32k_1^2(1-k_1^2)K^3(k_1)}, \quad f_1 = 4p_1^2 - \cos q_1 = 14k_1^2 \operatorname{cn}^2 u + 2k_1^2 - 1$$

$$f_2 = \frac{\partial}{\partial I}(p_1^2 + 2\cos q_1) = \frac{\pi}{(1-k_1^2)K(k_1)}[1 - k_1^2 + 2\operatorname{sn} u \operatorname{dn} u(\operatorname{cn} u \operatorname{zn} u - \operatorname{sn} u \operatorname{dn} u)]$$

The quantity u is defined in (2.4), and k_1 corresponds to the unperturbed motion. The dots in (4.1) denote terms of higher powers than five in $q_2, p_2, |r_1|^{1/2}$.

Proof of orbital stability when $\beta < \pi/2$. When $\beta < \pi/2$ we have $0 < k_1 < \sqrt{2}/2$. By (3.12) the quantity a in characteristic equation (3.9) satisfies the inequality $|a| < 1$, and consequently, there is stability in the linear approximation. The elements $x_{ij}(w)$ of the matrix of the fundamental solutions of system (3.6) will be bounded functions.

Lemma 1. If we make the replacement of variables $q_2, p_2 \to \tilde{q}_2, \tilde{p}_2$ in accordance with the formulae

$$q_2 = x_{11}(w)\tilde{q}_2 + x_{12}(w)\tilde{p}_2, \quad p_2 = x_{21}(w)\tilde{q}_2 + x_{22}(w)\tilde{p}_2 \tag{4.3}$$

the Hamiltonian $\tilde{h}_2$ of the converted system (3.2) will be identically equal to zero, and the function (3.5) will become

$$\tilde{k}_2 = 2(2k_1^2 - 1)\tilde{q}_2^2 - \tilde{p}_2^2 \tag{4.4}$$

The first assertion of the lemma follows immediately from the theory of variation of the arbitrary constants in systems of Hamilton differential equations [1, Section 187]. In order to show the correctness of the second assertion, we note that the quadratic form $\tilde{k}_2$, obtained from (3.5) by making replacement (4.3), is the integral of the converted system. But since $\tilde{h}_2 = 0$, the coefficients of the form k_2 should be constant quantities. Substituting (4.3) into the right-hand side of (3.5) and putting $w = 0$, we obtain expression (4.4) for $\tilde{k}_2$.

Suppose $S(q_2, \tilde{p}_2, w)$ is the generating function of canonical univalent transformation (4.3). In the equations of the perturbed motion, defined by Hamiltonian (4.1), we make the canonical univalent replacement of variables $q_2, p_2, r_1, w \to \tilde{q}_2, \tilde{p}_2, \tilde{r}_1, \tilde{w}$, specifying it using the generating function $\tilde{r}_1 w + S$. Then

$$r_1 = \tilde{r}_1 + \frac{\partial S}{\partial w} = \tilde{r}_1 - \frac{1}{4\omega_1}[p_2^2 + (p_1^2 + 2\cos q_1)q_2^2], \quad w = \tilde{w} \tag{4.5}$$

while q_2 and p_2 must be replaced in accordance with formulae (4.3).

Replacement (4.3), (4.5) cancels the quadratic form with respect to $\tilde{q}_2, \tilde{p}_2$ in the Hamiltonian, and reduces it to the form

$$\tilde{H} = \tilde{r}_1 + \tilde{H}_4 + \ldots \tag{4.6}$$

The function $\tilde{H}_4$ in (4.6) is obtained from (4.2) if we replace q_2, p_2 and r_1 in the latter in accordance with formulae (4.3) and (4.5).

The Kovalevskaya integral (3.4) can then be written in the new variables in the form

$$\tilde{K} = \tilde{k}_2 + \tilde{k}_4 + \ldots + \tilde{k}_n + \ldots = \text{const} \tag{4.7}$$

where $\tilde{k}_n$ is a form of the power n with respect to $\tilde{q}_2, \tilde{p}_2, |\tilde{r}_1|^{1/2}$, and $\tilde{k}_2$ is given by (4.4).

The problem of the orbital stability of the oscillations of the body is equivalent to the problem of their stability with respect to the variables $q_2, \tilde{p}_2, \tilde{r}_1$.

To prove the stability when $0 < k_1 < \sqrt{2}/2$ we consider the function $V = \tilde{H}^2 + \tilde{K}^2$. Since $\tilde{H}$ and $\tilde{K}$ are integrals of the equations of the perturbed motion, we have $dV/d\tau = 0$. But, in view of the fact that the function (4.4) is negative-definite with respect to the variables $\bar{q}_2, \bar{p}_2$, the system of equations $\bar{H} = 0$, $\bar{K} = 0$ for small $\bar{q}_2, \bar{p}_2, \bar{r}_1$ has only the zero solution $\bar{q}_2 = \bar{p}_2 = \bar{r}_1 = 0$. Hence, the function V is positive-definite with respect $\bar{q}_2, \bar{p}_2, \bar{r}_1$ and, consequently [5], the oscillations of the rigid body with an amplitude not exceeding $\pi/2$ are orbitally stable.

The stability for the critical value of the oscillation amplitude ($\beta = \pi/2$). When $\beta = \pi/2$ in Hamiltonian (4.1) of the perturbed motion and formula (4.2) we have

$$h_2 = \frac{K}{2\pi}(p_2^2 + 4\,\mathrm{cn}^2\,u\,q_2^2), \quad u = \frac{2K}{\pi}w \tag{4.8}$$

$$f_0 = -\frac{\pi^2(2E-K)}{16K^3}, \quad f_1 = 7\,\mathrm{cn}^2\,u, \quad f_2 = \frac{\pi}{K}[1 + 4\,\mathrm{sn}\,u\,\mathrm{dn}\,u(\mathrm{cn}\,u\,\mathrm{zn}\,u - \mathrm{sn}\,u\,\mathrm{dn}\,u)] \tag{4.9}$$

Here and below the modulus k_1 of the elliptic functions and integrals is equal to $\sqrt{2}/2$.

It turned out that when $k_1 = \sqrt{2}/2$ the matrix of the fundamental solutions of system (3.6) can be written in explicit form

$$\begin{aligned} x_{11} &= \mathrm{cn}u, \qquad x_{12} = \mathrm{sn}u\mathrm{dn}u - \mathrm{cn}u \quad F(u) \\ x_{21} &= -2\mathrm{sn}u\mathrm{dn}u, \qquad x_{22} = \mathrm{cn}^3u + 2\mathrm{sn}u\mathrm{dn}uF(u) \\ F(u) &= \mathrm{zn}\,u + \frac{2E-K}{2K}u \end{aligned} \tag{4.10}$$

When $w = \pi$ we have

$$\mathbf{X}(\pi) = \begin{Vmatrix} -1 & x_{12}(\pi) \\ 0 & -1 \end{Vmatrix}, \quad x_{12}(\pi) = 2E - K$$

The roots of characteristic equation (3.9) are multiple: $\rho_1 = \rho_2 = -1$, while the matrix $X(\pi)$ cannot be reduced to diagonal form. Hence, within the framework of the linear equations of perturbed motion, orbital instability occurs. We will show, however, that in the complete non-linear system of equations of perturbed motion the oscillations of a rigid body with amplitude equal to $\pi/2$ are orbitally stable. To prove this we will use the results obtained previously in [6]. To use them the Hamiltonian of the perturbed motion (4.1) must be reduced to normal form, from the coefficients of which we can judge the stability or instability.

In accordance with the algorithm from [6] we will first normalize the linear system with Hamiltonian (4.8) and the independent variable w. To do this we make the replacements $q_2, p_2 \to q_2^*, p_2^*$ in accordance with the formulae

$$q_2 = n_{11}q_2^* + n_{12}p_2^*, \quad p_2 = n_{21}q_2^* + n_{22}p_2^* \tag{4.11}$$

$$n_{11} = cx_{11}, \quad n_{12} = cx_{11}w + c^{-1}x_{12}$$

$$n_{21} = cx_{21}, \quad n_{22} = cx_{21}w + c^{-1}x_{22} \tag{4.12}$$

$$c = [x_{12}(\pi)/\pi]^{1/2} = [(2E-K)/\pi]^{1/2}$$

The quantities x_{ij} in (4.12) are the functions $x_{ij}(w)$ from (4.10).

Replacement (4.11) is univalent and canonical and has a period 2π with respect to w. The Hamiltonian $h_2^* = -\frac{1}{2}p_2^{*2}$ corresponds to the converted linear equations (3.2).

Supplementing Eqs (4.11) with the further two relations

$$r_1 = r_1^* - \frac{1}{2}p_2^{*2} - \frac{K}{2\pi}[(n_{21}q_2^* + n_{22}p_2^*)^2 + 4\,\mathrm{cn}^2\,u(n_{11}q_2^* + n_{12}p_2^*)^2], \quad w = w^* \tag{4.13}$$

we obtain the canonical univalent transformation $q_2, p_2, r_1, w \to q_2^*, p_2^*, r_1^*, w^*$ with respect to all four phase variables.

In the new variables, the Hamiltonian of the perturbed motion takes the form

$$H^* = r_1^* - \tfrac{1}{2} p_2^{*2} + H_4^* + \dots \tag{4.14}$$

Hamiltonian (4.14) has a period of 2π with respect to w. By the previous discussion in [6], when solving the stability problem, of the coefficients of the fourth-degree form H_4^* with respect to $q_2^*, p_2^*, |r_1^*|^{1/2}$, as a rule it is sufficient to know only the coefficient g_{40} for q_2^{*4}. We obtain the following expression for it from relations (4.1) and (4.8)–(4.13)

$$g_{40} = \frac{(2E-K)^2}{12\pi^3}\{7K\,\mathrm{cn}^6 u - 6(2E-K)(\mathrm{sn}^2 u\,\mathrm{dn}^2 u + \mathrm{cn}^4 u)^2 -$$

$$-12K[1 + 4\,\mathrm{sn}\,u\,\mathrm{dn}\,u(\mathrm{cn}\,u\,\mathrm{zn}\,u - \mathrm{sn}\,u\,\mathrm{dn}\,u)](\mathrm{sn}^2 u\,\mathrm{dn}^2 u + \mathrm{cn}^4 u)\mathrm{cn}^2 u\} \tag{4.15}$$

Using the canonical replacement of variables $q_2^*, p_2^*, r_1^*, w \to \xi_2, \eta_2, \eta_1, \xi_1$ in Hamiltonian (4.14), we can normalize the fourth-degree terms [6], and it takes the form

$$\Gamma = \eta_1 - \tfrac{1}{2}\eta_2^2 + h_{40}\xi_2^4 + h_{20}\xi_2^2\eta_1 + h_{00}\eta_1^2 + O_6$$

where we have denoted the set of terms higher than the fifth power in ξ_2, η_2, $|\eta_1|^{1/2}$, which are 2π-periodic in ξ_1, by O_6. The coefficients h_{ji} are constant quantities, and $h_{40} = \langle g_{40}\rangle$, where $\langle g_{40}\rangle$ is the average value of the function (4.15) during its period 2π with respect to w.

The periodic motion investigated is orbitally stable if the signs of the coefficient h_{40} and of the coefficient of η_2^2 in the normalized Hamiltonian are the same, and unstable if these signs are opposite [6].

Using the identities which relate elliptic functions [3], their evenness and oddness properties, and employing integration by parts, the expression for the coefficients h_{40} can be represented in the following form

$$h_{40} = \frac{K(2E-K)^2}{24\pi^3}(9\langle \mathrm{cn}^2 u\rangle - 4\langle \mathrm{cn}^6 u\rangle - 27\langle \mathrm{cn}^{10} u\rangle)$$

where $\langle \mathrm{cn}^m u\rangle$ is the average value of the function $\mathrm{cn}^m u$ over u during its period.

But when $k_1 = \sqrt{2}/2$ we have $K = 1.85407$ and $E = 1.35064$, while

$$\langle \mathrm{cn}^2 u\rangle = 0.45695,\quad \langle \mathrm{cn}^6 u\rangle = 0.27417,\quad \langle \mathrm{cn}^{10} u\rangle = 0.21324$$

Hence, $h_{40} = -0.0049 < 0$, and consequently, oscillations of amplitude equal to $\pi/2$ are orbitally stable.

The proof of Theorem 1 on the stability of plane oscillations and rotations of a Kovalevskaya top around an axis of dynamic symmetry is therefore concluded.

Remark. In the Kovalevskaya case there are also plane oscillations and rotations of the body about the Oy axis, lying in the equatorial plane of the ellipsoid of inertia for the fixed point O. Here the Oy axis occupies a fixed horizontal position, while the Ox axis, on which the centre of gravity of the body lies, and the axis of dynamic symmetry Oz move in a fixed vertical plane passing through the point O. Research showed that these conclusions regarding the stability of these motions are completely analogous to the conclusions which have been reached above when analysing the stability of oscillations and rotations about an axis of dynamic symmetry: rotations and oscillations with amplitude exceed $\pi/2$ are orbitally unstable, while oscillations with amplitudes no greater than $\pi/2$ are orbitally stable. Note that the stability of oscillations with amplitude less than $\pi/2$ was proved previously in [7].

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